ON ALGEBRAIC INTEGRALS IN THE PROBLEM OF MOTION OF A HEAVY GYROSTAT FIXED AT ONE POINT

(OB ALGEBRAICHESKIKH INTEGRALAKH V ZADACHE O DVIZHENII TIAZHOLOGO GIROSTATA, ZAKREPLENNOGO V ODNOI TOCHKE)

PMM Vol.28, № 3, 1964, pp.516-520

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(Received November 2, 1963)

1. The equations of motion of a heavy gyrostat fixed at one point, as is known [1], are of the form

$$A\frac{dp}{dt}+(C-B)\,qr+m_3r-m_2r=y_0'\gamma_3-z_0'\gamma_2,\qquad \frac{d\gamma_1}{dt}=r\gamma_2-q\gamma_3\,.\qquad (x_0'=Mgx_0)\qquad(1.1)$$

$$B\,\frac{dq}{dt}+(A-C)\,pr+m_1r-m_3p=z_0'\gamma_1-x_1'\gamma_3,\qquad \frac{d\gamma_2}{dt}=p\gamma_3-r\gamma_1$$

$$C\,\frac{dr}{dt}+(B-A)\,pq+m_2p-m_1q=\varkappa_0'\gamma_2-y_0'\gamma_1,\qquad \frac{d\gamma_3}{dt}=q\gamma_1-p\gamma_2$$

The system (1.1) does not contain time explicitly, has its last Jacobi multiplier equal to unity, and admits of the algebraic integrals

$$Ap^2+Bq^2+Cr^2-2\,(x_0'\gamma_1+y_0'\gamma_2+z_0'\gamma_3)=h_1$$

$$(Ap+m_1)\,\gamma_1+(Bq+m_2)\,\gamma_2+(Cr+m_3)\,\gamma_3=h_2,\qquad \gamma_1^2+\gamma_2^2+\gamma_3^2=1\qquad(1.2)$$

As was shown in [1 and 2], the system of equations (1.1) reduces to quadratures if $x_0'=y_0'=z_0'=0$, when it admits of the fourth algebraic integral

$$(Ap+m_1)^2+(Bq+m_2)^2+(Cr+m_3)^2=L^2$$

and in the case $A=B$, $x_0'=y_0'=0$, $m_1=m_2=0$, when it has the integral $r=r_0$.

We formulate the problem of finding general conditions for the existence of new algebraic integrals of the system (1.1) which are independent of the clasical integrals (1.2).

2. We shall show, first of all, that if Poincaré's theorem [3 to 5] holds, then for the existence of a new algebraic integral it is necessary that the ellipsoid of inertia with respect to the fixed point be an ellipsoid of revolution. We replace

$$p,\ q,\ r,\ \gamma_1,\ \gamma_2,\ \gamma_3 t\quad \textbf{by}\quad \lambda^{-2}p,\ \lambda^{-2}q,\ \lambda^{-2}r,\ \lambda^{-4}\gamma_1,\ \lambda^{-4}\gamma_2,\ \lambda^{-4}\gamma_3,\ \lambda^2t+t_0$$

where λ is an arbitrary parameter, and we introduce the new variables

$$y_1=\sqrt{A\,(A-C)}\;p+i\sqrt{B\,(B-C)}\;q,\qquad y_2=\sqrt{A\,(A-C)}\;p-i\sqrt{B\,(B-C)}\;q$$

$$z_1=\gamma_1+i\gamma_2,\qquad z_2=\gamma_1-i\gamma_2$$

As was done in [4 and 5], we replace

$$y_1,\ y_2,\ z_1,\ z_2,\ \gamma_3,\ t \text{ by } \lambda y_1,\ y_2,\ r,\ \lambda z_1,\ \lambda z_2,\ \lambda\gamma_3,\ -it$$

Utilizing the integrals (1.2) for Equations (1.1) in new variables, following [5], we can prove Poincaré's theorem: if the ellipsoid of inertia is not an ellipsoid of revolution, then for arbitrary initial conditions, except for the case $x_0' = y_0' = z_0' = 0$, there cannot exist any new algebraic integral of the system (1.1).

3. We shall prove that if $x_0'^2 + y_0'^2 + z_0'^2 \neq 0$ and $m_1^2 + m_2^2 + m'^2 \neq 0$, the the fourth algebraic integral is possible only in the case of a gyrostatic analog of Lagrange's case [2] defined by conditions

$$A = Bx_0' = y_0' = 0,\ m_1 = m_2 = 0$$

We replace

$$p,\ q,\ r,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ t \text{ by } \lambda^{-5}p,\ \lambda^{-5}q,\ \lambda^{-5}r,\ \lambda^{-10}\gamma_1,\ \lambda^{-10}\gamma_2,\ \lambda^{-10}\gamma_3,\ \lambda^5 it + t_0$$

and introduce in Equation (1.1) the variables

$$y_1 = p + iq, \quad y_2 = p - iq, \quad z_1 = \gamma_1 + i\gamma_2, \quad z_2 = \gamma_1 - i\gamma_2$$
$$m_1' = m_1 + im_2, \quad m_2' = m_1 - im_2$$

Let us assume that, in accordance with Poincaré's theorem, $A = B$; then, for an appropriate choice of the coordinate system, we find $y_0' = 0$. The system (1.1) becomes the system

$$A\frac{dy_1}{dt} = (C-A)\,ry_1 + z_0'z_1 - x_0'\gamma_3 + \lambda^5\ (m_3y_1 - m_1'r), \quad \frac{dz_1}{dt} = y_1\gamma_3 - rz_1$$
$$A\frac{dy}{dt} = (A-C)\ ry_2 - z_0'z_2 + x_0'\gamma_3 + \lambda^5\ (m_2'r - m_3y_2), \qquad \frac{dz_2}{dt^2} = rz_2 - y_2\gamma_3 \qquad (3.1)$$
$$2C\frac{dr}{dt} = x_0'\ (z_2 - z_1) + \lambda^5\ (m_2'y_1 - m_1'y_2), \qquad 2\ \frac{d\gamma}{dt^3} = y_2z_1 - y_1z_2$$

For the system (3.1) we have the following first integrals:

$$Ay_1y_2 + Cr^2 - x_0'\ (z_1 + z_2) - 2z_0'\gamma_3 = h_1 \qquad (3.2)$$
$$A\ (y_1z_2 + z_1y_2) + 2Cr\gamma_3 + \lambda^5\ (m_1'z_2 + m_2'z_1 + 2m_3\gamma_3) = h_2,\ \ z_1z_2 + \gamma_3^2 = h_3$$

If we again introduce an arbitrary parameter by replacing

$$y_1,\ y_2,\ r,\ z_1,\ z_2,\ \gamma_3,\ t \text{ by } \lambda^2y_1,\ \lambda y_2,\ \lambda r,\ \lambda^3z_1,\ \lambda^2z_2,\ \lambda^3\gamma_3,\ \lambda^{-1}t + t_0$$

then the system (3.1) and its first integrals (3.2) assume the form

$$A\frac{dy_1}{dt} = -\ (A-C)\ ry_1 + z_0'z_1 - x_0'\gamma_3 - \lambda^3\ (m_1'r - \lambda m_3y_1), \qquad \frac{dz_1}{dt} = -\ rz_1 + \lambda y_1\gamma_3$$
$$A\frac{dy_2}{dt} = \quad (A-C)\ ry_2 - z_0'z_2 + \lambda x_0'\gamma_3 + \lambda^4\ (m_2'r - m_3y_2), \qquad \frac{dz_2}{dt} = rz_2 - \lambda y_2\gamma_3$$
$$2C\ \frac{dr}{dt} = x_0'\ (z_2 - \lambda z_1) - \lambda^4\ (m_1'y_2 - \lambda m_2'y_1), \qquad 2\frac{d\gamma_3}{dt} = y_2z_1 - y_1z_2 \qquad (3.3)$$
$$Cr^2 - x_0'z_2 + \lambda\ (Ay_1y_2 - x_0'z_1 - 2z_0'\gamma_3) = h_1$$
$$A\ (y_1z_2 + z_1y_2) + 2Cr\gamma_3 + \lambda^3\ (m_1'z_2 + \lambda m_2'z_1 + 2\lambda m_3\gamma_3) = h_2,\ z_1z_2 + \lambda^2\gamma_3^2 = h_3 \ (3.4)$$

Husson showed that if $m_1' = m_2' = m_3 = 0$, the fourth algebraic integral exists in Lagrange's case ($A = B$, $x_0' = y_0' = 0$) and Kowalewski's case ($A = B = 2C$, $y_0' = z_0' = 0$) ; the proof utilized the first three terms of the series expansion of the general integral of the resulting system of equations in powers of the parameter λ, which was assumed to be small. It was also taken into account that the right-hand sides of the resulting system of equations and its first integrals were polynomials in y_1, y_2, r, z_1, z_2, γ_3 and λ. It may be noted that the first three terms of the series expansions in powers of λ of the general untegral of the system (3.3) and integrals (3.4) are polynomials in y_1, y_2, r, z_1, z_2, γ_3 and λ and are independent of m_1', m_2' and m_3. For this reason Husson's result should be regarded as one of the necessary conditions for the existence of a new fourth algebraic integral of the system (1.1). Thus, the problem reduces to finding new conditions in addition to those which hold for Kowalewski's and Lagrange's cases.

4. Let us find these conditions for Kowalewski case. We set

$$A = B = 2C_1y_0' = z_0' = 0, c = x_0C^{-1}, \ \mu_i = m_i'C^{-1} \ (i = 1, 2) \ m_3' = m_3C^{-1}$$

and, replacing in (1.1) the quantities

$$p, q, r, \gamma_1, \gamma_2, \gamma_3, t \text{ на } \lambda^{-1/2}p, \lambda^{-1/2}q, \lambda^{-1/2}r, \lambda^{-1}\gamma_1, \lambda^{-1}\gamma_2, \lambda^{-1}\gamma_3, \lambda^{1/2}t + t_0$$

we introduce

$$y_1 = p + qi, \ y_2 = p - qi, \ z_1 = \gamma_1 + \gamma_2 = \gamma_1 - i\gamma_2, \ m_1' = m_1 + im_2, \ m_2 = m_1 - im_2$$

By replacing

$$y_1, y_2, r, z_1, z_2, \gamma_3, t \quad \text{by} \quad \lambda^{1/2}y_1, \lambda^{-1/2}y_2, \lambda^{-1/2}r, z_1, \lambda^{-1}z_2, \gamma_3; \ i\lambda^{1/2}t + t_0$$

as indicated in [6], we obtain instead of (1.1) the system of equations

$$\begin{aligned} 2\frac{dy_1}{dt} &= -ry_1 - c\gamma_3 + \lambda m_3'y_1 - \mu_1 r, & \frac{dz_1}{dt} &= -rz_1 + \lambda y_1\gamma_3 \\ 2\frac{dy_2}{dt} &= ry_2 + \lambda c\gamma_3 - \lambda m_3'y_2 + \lambda\mu_2 r, & \frac{dz_2}{dt} &= rz_2 - \lambda y_2\gamma_3 \\ 2\frac{dr}{dt} &= c(z_2 - \lambda z_1) + \lambda^2\mu_2 y_1 - \lambda\mu_1 y_2, & 2\frac{d\gamma_3}{dt} &= y_2z_1 - y_1z_2 \end{aligned} \tag{4.1}$$

For the system (4.1) the following first algebraic integrals exist:

$$\begin{gathered} r^2 - cz_2 + \lambda(2y_1y_2 - cz_1) = h_1 \\ 2y_1z_2 + 2zy_2 + 2r\gamma_3 + \mu_1z_2 + \lambda(\mu_2z_1 + 2m_3'\gamma_3) = h_2, \qquad z_1z_2 + \lambda\gamma_3^2 = h_3 \end{gathered} \tag{4.2}$$

If (4.1) has a fourth algebraic integral, then according to [4], the system

$$\frac{dy_2}{dr} = \frac{ry_2 + \lambda c\gamma_3 - \lambda m'_3y_2 + \lambda\mu_2 r}{c(z_2 - \lambda z_1) - \lambda\mu_1y_2 + \lambda^2\mu_2y_1}, \qquad \frac{d\gamma_3}{dr} = \frac{y_2z_1 - y_1z_2}{c(z_2 - \lambda z_1) - \lambda\mu_1y_2 + \lambda^2\mu_2y_1} \tag{4.3}$$

in which y_1, z_1 and z_2 are expressed as functions of y_2, r, γ_3, h_1, h_2 and h_3 from (4.2), has the algebraic integral $F(y_2, \gamma_3, r, h_1, h_2, h_3)$ = const This integral can be expanded into a series in powers of $\lambda^{1/n}$ in a neighborhood of $\lambda = 0$ (where n is an integer):

$$F_0(y_2, \gamma_3, r) + \lambda^{1/n}F_1(y_2, \gamma_3, r) + \ldots + \lambda F_n(y_2, \gamma_3, r) + \ldots = \text{const} \tag{4.4}$$

with coefficients which are algebraic functions of their arguments. The quantity F_0 necessarily depends on at least one of the quantities y_1 and γ_3. If λ is sufficiently small, the general solution of the system may be expanded into series [5] in integer powers

$$\begin{gathered} y_1 = y_1^{(0)} + \lambda y_1^{(1)} + \ldots, \quad y_2 = y_2^{(0)} + \lambda y_2^{(1)} + \ldots, \quad \gamma_3 = \gamma_3^{(0)} + \lambda\gamma_3^{(1)} + \ldots \\ z_1 = z_1^{(0)} + \lambda z_1^{(1)} + \ldots, \qquad z_2 = z_2^{(0)} + \lambda z_2^{(1)} + \ldots \end{gathered}$$

Substituting this expansion into the integral (4.4), we obtain

$$F_0 (y_2^{(0)}, \gamma_3^{(0)}, r) + \lambda^{1/n} F_1 (y_2^{(0)}, \gamma_3^{(0)}, r) + \ldots$$
$$\ldots + \lambda \left[F_n + y_2^{(1)} \frac{\partial F_0}{\partial y_2^{(0)}} + \gamma_3^{(1)} \frac{\partial F_0}{\partial \gamma_3^{(0)}} \right] + \ldots + \lambda^2 (\ ) + \ldots = \text{const}$$

As was shown in [4 and 5], this relation makes it possible to represent the first integrals, which a system of the form

$$\frac{dy_2^{(0)}}{dr} = \frac{1}{cz_2^{(0)}} r y_2^{(0)}, \qquad \frac{d\gamma_3^{(0)}}{dr} = \frac{1}{cz_2^{(0)}} [y_2^{(0)} z_1^{(0)} - y_1^{(0)} z_2^{(0)}] \tag{4.5}$$

$$\frac{dy_2^{(1)}}{dr} = \frac{1}{cz_2^{(0)}} \left[r y_2^{(1)} + c\gamma_3^{(0)} - m_3' y_2^{(0)} + \mu_2 r - \frac{1}{cz_2^{(0)}} [c(z_2^{(1)} - z_1^{(0)}) - \mu_1 y_2^{(0)}] r y_2^{(0)} \right]$$
$$\frac{d\gamma_3^{(1)}}{dr} = \frac{1}{cz_2^{(0)}} \Big[y_2^{(1)} z_1^{(0)} + y_2^{(0)} z_1^{(1)} - y_1^{(0)} z_2^{(1)} - y_1^{(1)} z_2^{(0)} - \tag{4.6}$$
$$- \frac{1}{cz_2^{(0)}} [c(z_2^{(1)} - z_1^{(0)}) - \mu_1 y_2^{(0)}] \ (y_2^{(0)} z_1^{(0)} - y_1^{(0)} z_2^{(0)}) \Big]$$

must have, in the following manner:

$$F_0 (y_2^{(0)}, \gamma_3^{(0)}, r) = \text{const} \tag{4.7}$$

$$F_n (y_2^{(0)}, \gamma_3^{(0)}, r) + y_2^{(1)} \frac{\partial F_0}{\partial y_2^{(0)}} + \gamma_3^{(1)} \frac{\partial F_0}{\partial \gamma^{(0)}} = \text{const} \tag{4.8}$$

Let us consider the particular solution of the system (4.3), setting

$$h_1 = a^2, \quad h_2 = \lambda^2, \quad h_3 = \lambda^2$$

Then from (4.2) we find

$$r^2 - cz_2 = a^2, \qquad 2y_1^{(0)} z_2^{(0)} + 2z_1^{(0)} y_2^{(0)} + 2r\gamma_3^{(0)} + \mu_1 z_2^{(0)} = 0, \qquad z_1^{(0)} z_2^{(0)} = 0$$

It follows from this that

$$z_2^{(0)} = \frac{r^2 - a^2}{c}, \quad z_1^{(0)} = 0, \qquad y_1^{(0)} = - \frac{cr}{r^2 - a^2} \gamma_3^{(0)} - \frac{\mu_1}{2}$$

Then from (4.5) for the determination of $y_2^{(0)}$ and $\gamma_3^{(0)}$ we have

$$\frac{dy_2^{(0)}}{dr^2} = \frac{r}{r^2 - a^2} y_2^{(0)}, \qquad \frac{d\gamma_3^{(0)}}{dr} = \frac{r}{r^2 - a^2} \gamma_3^{(0)} + \frac{\mu_1}{2c} \tag{4.9}$$

The equations (4.9) will be satisfied by the following particular solutions:

$$y_2^{(0)} = 0, \quad \gamma_3^{(0)} = \frac{\mu_1}{2c} \sqrt{r^2 - a^2} \ln (r + \sqrt{r^2 - a^2}) \tag{4.10}$$

Since Equation (4.7) determines the algebraic relationship between the above-mentioned arguments, then in order to eliminate the contradiction which is evident in an examination of (4.7) and (4.10), it must be assumed that either F_0 is independent of $\gamma_3^{(0)}$ or $\mu_1 = 0$.

Assuming the first case, we replace in (1.1)

$$p, q, r, \gamma_1, \gamma_2, \gamma_3, t \quad \text{by} \quad \lambda^{-1} p, \lambda^{-1} q, \lambda^{-1} r, \lambda^{-2} \gamma_1, \lambda^{-2} \gamma_2, \lambda^{-2} \gamma_3 \ \lambda t + t_0$$

respectively, and we write the equations, retaining the earlier notation and quantities, and replacing

$$y_1, y_2, r, z_1, z_2, \gamma_3, t \ \text{by} \ \lambda y_1, \lambda y_2, r, \lambda^2 z_1, z_2, \lambda \gamma_3, \lambda^2 t + t_0$$

In a manner analogous to the foregoing, the problem reduces to the study of a system of the form

$$\frac{dy_2}{dr} = \frac{ry_2 + \mu_2 r + c\gamma_3 - \lambda m_3' y_2}{c(z_2 - \lambda^2 z_1) - \lambda^2\mu_1 y_2 + \lambda^2\mu_2 y_1}, \quad \frac{d\gamma_3}{dr} = \frac{-y_1 z_2 + \lambda^2 y_1 y_2}{c(z_2 - \lambda^2 z_1) - \lambda^2\mu_1 y_2 + \lambda^2\mu_2 y_1} \qquad (4.11)$$

in which y_1, z_1 and z_2 must be replaced by functions of y_2, r, γ_3, h_1, h_2 and h_3 from Equations

$$r^2 - cz_2 - \lambda^2 cz_1 + 2\lambda^2 y_1 y_2 = h_1, \quad 2r\gamma_3 + \mu_1 z_2 + 2y_1 z_2 + 2\lambda m_3'\gamma_3 + 2\lambda^2 y_2 z_1 + \lambda^2\mu_2 z_1 = h_2$$
$$z_1 z_2 + \gamma_3^2 = h_3 \qquad (4.12)$$

Let us consider the particular solution of the system (4.11) determined by the following values of the srbitrary constants of (4.12):

$$h_1 = a^2, \quad h_2 = \lambda^2, \quad h_3 = \lambda^2 \qquad (4.13)$$

Then, utilizing equations (4.12) and (4.13), we pass from the system (4.11) to the system

$$\frac{dy_2}{dr} = \frac{ry_2^{(0)} + \mu_2 r + c\gamma_3^{(0)}}{cz_2^{(0)}}, \quad \frac{d\gamma_3^{(0)}}{dr} = -\frac{y_1^{(0)}}{c} \qquad (4.14)$$

The system (4.14) will be satisfied by the particular solutions

$$y_2^{(1)} = \mu_2\varphi(r^2 - a^2) + \frac{\mu_1}{2}\int_{r_0}^{r} \frac{\ln(r + \sqrt{r^2 - a^2})}{r^2 - a^2}\,dr$$

$$\gamma_3^{(0)} = \frac{\mu_1}{2c}\sqrt{r^2 - a^2}\ln(r + \sqrt{r^2 - a^2})$$

where $\varphi(r^2 - a^2)$ is an algebraic function of the above argument.

If $\mu_1 \neq 0$, the function $F_0(y_2^{(0)}, r)$ in a manner similar to the foregoing, must be dependent of $y_2^{(0)}$ which contradicts the property of

$$F_0(y_2^{(0)}, \gamma_3^{(0)}, r)$$

and we must assume from [4 and 5], that $\mu_1 = 0$ is a necessary condition for the existence of a fourth algebraic integral of the problem under discussion. If $\mu_1 = 0$, then $\mu_2 = 0$, which is obvious. We shall show that $m_3' = 0$ as well. Introducing new variables into (1.1), as in the previous case, and replacing

$$y_1, y_2, r, z_1, z_2, \gamma_3, t \text{ by } \lambda y_1, y_2, r, \lambda z_1, z_2, \lambda\gamma_3, \lambda t + t_0$$

we reduce the problem to the study of a system of the form

$$\frac{dy_2}{dr} = \frac{ry_2 + \lambda c\gamma_3 - \lambda m_3' y_2}{c(z_2 - \lambda z_1)}, \quad \frac{d\gamma_3}{dr} = \frac{y_2 z_1 - y_1 z_2}{c(z_2 - \lambda z_1)} \qquad (4.15)$$

In a manner analogous to the above, it can be shown that a necessary condition for the existence of an algebraic integral of the system (4.15) will be $m_3' = 0$.

Thus, it can be asserted that the first set of necessary conditions for the existence of a new fourth algebraic integral of the system (1.1) can be written in the form $A = B = 2C$, $y_0' = z_0' = 0$, $m_1 = m_2 = m_3 = 0$.

5. We shall now prove the assertion made in Section 3. In equations (1.1) we replace

$$p, q, r, \gamma_1, \gamma_2, \gamma_3, t \text{ by } \lambda^{-1}p, \lambda^{-1}q, \lambda^{-1}r, \lambda^{-2}\gamma_1, \lambda^{-2}\gamma_2, \lambda^{-2}\gamma_3, \lambda t + t_0$$

Then for y_1, y_2, r and γ_3 we obtain equations of the form

$$\frac{dy_1}{dt} = (m-1)\, y_1 r + z_0 z_1 + \lambda\,(m_3 y_1 - m_1 r), \qquad \frac{dr}{dt} = \lambda \frac{m_2 y_1 - m_1 y_2}{2m} \qquad (5.1)$$

$$\frac{dy_2}{dt} = (1-m)\, y_2 r - z_0 z_2 + \lambda\,(m_2 r - m_3 y_2), \qquad \frac{d\gamma_3}{dt} = \frac{y_2 z_1 - y_1 z_2}{2}$$

where $m = CA^{-1}$, while $m_1'A^{-1}$ and $z_0'A^{-1}$ are replaced, respectively, by m_1 and z_0 .

Directing the axis Ox along the directional vector of the equatorial component of $\mathbf{m}$, we find $m_1 = m_2$. Let us consider the quantities

$$\xi = \lambda^{-1}(y_1 + y_2), \quad \eta = y_1 - y_2, \quad \alpha = z_1 + z_2, \quad \beta = \lambda^{-2}(z_1 - z_2), \quad u = \lambda^{-1} r$$

In order to determine these we have Equations

$$\frac{d\xi}{du} = 2m \frac{(m-1)[u + m_3/(m-1)]\eta + \lambda z_0 \beta}{m_1 \eta}, \qquad \frac{d\gamma_3}{du} = \frac{m}{2}\, \frac{-\eta\alpha + \lambda^3 \xi\alpha}{m_1 \eta} \qquad (5.2)$$

In a manner analogous to the above, in the new variables we obtain a system of intergrals of Equations (1.1) in the form

$$-\eta^2 - 8 z_0 \gamma_3 + \lambda^2(\xi^2 + 4mu^2) = h_1, \quad \xi\alpha + 2m_1\alpha + 4m\,(u + m_3/m)\gamma_3 - \lambda\eta\beta = h_2$$

$$\alpha^2 + 4\gamma_3^2 - \lambda^4\beta^2 = h_3 \qquad (5.3)$$

From the system (5.2) and the integrals (5.3) for ξ_0 and $\gamma_3^{(0)}$ we have Equations

$$\frac{d\xi_0}{du} = 2m \frac{m-1}{m_2}\left(u + \frac{m_3}{m-1}\right), \qquad \frac{d\gamma_3^{(0)}}{\sqrt{h_3 - 4\gamma_3^{(0)2}}} = -\frac{m}{2m_1}\, du \qquad (5.4)$$

which have the following particular solution :

$$\xi_0 = 2\frac{m(m-1)}{m_1}\int_{u_0}^{u}\left(u + \frac{m_3}{m-1}\right) du, \qquad \gamma_3^{(0)} = -\frac{\sqrt{h_3}}{2}\sin\left(\frac{m}{m_1}\, u\right)$$

In a manner analogous to [4 and 5], we conclude that if the fourth algebraic integral exists, it must depend only on ξ and u

Let us consider the particular solutions for the values $h_1 = h_2 = \lambda^2$, $h_3 = 4\lambda^2$. In this case, from (5.2) and (5.3) for determining ξ_1 and $\gamma_3^{(1)}$, we find Equations

$$\frac{d\xi_1}{du} = -\frac{m[\xi_0 + 2m(u + m_3/m) + 2m_1 i]\,\gamma_1 + i\gamma_0\xi_1}{2m_1\gamma_0}, \qquad \frac{d\gamma_3^{(1)}}{du} = -\frac{im}{m_1}\gamma_3^{(1)} \qquad (5.5)$$

which have the following particular solutions :

$$\xi_1 = \exp\left(\frac{-imu}{2m_1}\right) \qquad \gamma_3^{(1)} = 0$$

This indicates that the fourth algebraic integral is independent of ξ as well, but this contradicts the property of the integral, or else the reasoning does not hold, which is possible only if $m_1 = m_2 = 0$. In this case, as follows from [2], the classical integral $r = r_c$ exists, and we have completely proved the assertion of Section 3 that if $x_0'^2 + y_0'^2 + z_0'^2 \neq 0$ and $m_1^2 + m_2^2 + m_3^2 \neq 0$ a fourth algebraic integral is possible only when $A = B$, $x_0' = y_0' = 0$, $m_1 = m_2 = 0$.

The author expresses his gratitude to Iu.A. Arkhangel'skii and P.V. Miasnikov for their valuable advice.

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Translated by A.S.